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Linear integral equations

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INTRODUCTION

An equation of the form

(1) $$u(x) = f(x) + \lambda\int_a^b K(x,t)u(t)dt,$$

is said to be a linear integral equation of the second kind with a parameter. The functions $f(x)$ and $K(x,t)$, the limits a and b, and the parameter λ are known, and it is then desired to determine a function $u(x)$ such that (1) is satisfied for all x in the interval $[a,b]$. The function $K(x,t)$ is called the kernel of the equation, and is assumed to be real, continuous and not identically zero in the rectangle for which $a \leq x \leq b$, $a \leq t \leq b$. The function $f(x)$ is likewise assumed to be real, continuous and not identically zero in the interval $[a,b]$. The limits a and b and the parameter λ are constants.

The homogeneous linear integral equation is one in which $f(x) \equiv 0$ and thus is of the form

(2) $$u(x) = \lambda\int_a^b K(x,t)u(t)dt.$$

The linear integral equation of the second kind is a particular case of a more general equation, namely the linear integral equation of the third kind

(3) $$\Phi(x)u(x) = f(x) + \int_a^b K(x,t)u(t)dt,$$

for which $\Phi(x) \equiv 1$.

If in (3) $\Phi(x) \equiv 0$ we get the linear integral equation of the first kind,

$$(4) \qquad f(x) = \int_a^b K(x,t)u(t)dt.$$

Note that in all three cases the unknown u enters linearly and occurs under the definite integral; thus the name linear integral equation.

The purpose of this paper is to study the mathematical theory of the solution of linear integral equations. Such equations have numerous practical applications, particularly in the field of mechanical vibrations.

We will begin our discussion by studying Fredholm's solution of (1). This solution, which gives $u(x)$ as the ratio of two absolutely convergent power series in λ, holds for all values of λ, whereas certain other solutions hold only for restricted values of λ. Each of these series has an infinite radius of convergence. The coefficients in the numerator depend on x, but those of the denominator are independent of x. In the exceptional case for which the denominator vanishes, this method still furnishes a solution, provided that a solution exists.

To obtain his solution, Fredholm replaced the definite integral by the limit of a sum and was thus led to a system of n linear equations in n unknowns. We will constantly make use of this analogy between the linear integral equation and the system of linear equations in obtaining the solution of (1) under various conditions.

In Chapter II we approach the same problem from a somewhat different viewpoint, however, maintaining an analogy between a finite system of linear equations and the linear integral equation. We will first determine solutions of equations (1) and (2) for the case of degenerate kernels. These results will then be utilized to obtain solutions of (1) and (2) for arbitrary kernels, in which case the solutions are given as limits of sequences of solutions of equations of the types (1) and (2) with degenerate kernels.

CHAPTER I

THE FREDHOLM THEORY

1.1 DEFINITIONS AND SOME FUNDAMENTAL RELATIONS

Before we can produce a solution of (1) it will be necessary to define two power series in λ, and establish certain relations between them which will be fundamental in obtaining the desired solution. Also, we must develop an analogy between the integral equation and a system of linear equations.

To this end we first exhibit a sum whose limit is the definite integral in

$$u(x) = f(x) + \lambda\int_a^b K(x,t)u(t)dt. \tag{1-1}$$

Divide the interval [a,b] into n equal parts and call the points of division t_i, $i = 1,\ldots,n$. Now replace the definite integral in (1-1) by the sum of which it is the limit. We thus obtain:

$$u(x) = f(x) + \lambda[K(x,t_1)u(t_1)\Delta t + \cdot\cdot\cdot + K(x,t_n)u(t_n)\Delta t]. \tag{1-2}$$

Since this holds for all x in the interval, we can let $x = t_1, t_2,\ldots, t_n$, which produces the n linear equations in the n unknowns $u(t_i)$, $i = 1,\ldots,n$:

$$\begin{aligned}
u(t_1) - \lambda\Delta t[K(t_1,t_1)u(t_1) + \cdot\cdot\cdot + K(t_1,t_n)u(t_n)] &= f(t_1),\\
u(t_2) - \lambda\Delta t[K(t_2,t_1)u(t_2) + \cdot\cdot\cdot + K(t_2,t_n)u(t_n)] &= f(t_2),\\
\vdots \qquad\qquad\qquad & \quad \vdots\\
u(t_n) - \lambda\Delta t[K(t_n,t_1)u(t_n) + \cdot\cdot\cdot + K(t_n,t_n)u(t_n)] &= f(t_n).
\end{aligned} \tag{1-3}$$

Denoting by Δ the determinant of the coefficients of $u(t_i)$, and by Δ_{ij} the minor of the element in the i-th row, j-th column, we can solve for $u(t_k)$ by Cramer's Rule, provided $\Delta \neq 0$:

$$u(t_k) = \frac{\sum_{i=1}^{n} f(t_i)\Delta_{ik}}{\Delta}, \qquad k = 1,\ldots,n.$$

Consider now the limits of Δ, Δ_{kk}, Δ_{ij} as n is allowed to increase indefinitely.

i). Limit of Δ. Expanding Δ by considering all the possible ways in which to get λ^s, s = 0,1,...,n, without repetition, we obtain

$$\Delta = 1 - \lambda\sum_{i=1}^{n} K(t_i,t_i)\Delta t + \frac{\lambda^2}{2!}\sum_{i,j=1}^{n}\begin{vmatrix} K(t_i,t_i) & K(t_i,t_j) \\ K(t_j,t_i) & K(t_j,t_j)\end{vmatrix}(\Delta t)^2 +$$

$$+ \cdots + (-1)\frac{\lambda^q}{q!}\sum_{\substack{i_1=1\\ \vdots \\ i_q=1}}^{n}\begin{vmatrix} K(t_{i_1},t_{i_1}) & \cdots & K(t_{i_1},t_{i_q}) \\ \vdots & & \vdots \\ K(t_{q_1},t_{i_1}) & \cdots & K(t_{q_1},t_{q_1})\end{vmatrix}(\Delta t)^q +$$

$$+ \cdots + (-1)^n\lambda^n\begin{vmatrix} K(t_1,t_1) & \cdots & K(t_1,t_n) \\ \vdots & & \vdots \\ K(t_n,t_1) & \cdots & K(t_n,t_n)\end{vmatrix}(\Delta t)^n.$$

Therefore,

$$\lim_{n\to\infty} \Delta = 1 - \lambda \int_a^b K(t,t)dt + \frac{\lambda^2}{2!}\int_a^b\int_a^b \begin{vmatrix} K(t_1,t_1) & K(t_1,t_2) \\ K(t_2,t_1) & K(t_2,t_2) \end{vmatrix} dt_1 dt_2 +$$

$$+ \cdot \cdot \cdot + \frac{\lambda^q}{q!}\int_a^b \cdots \int_a^b \begin{vmatrix} K(t_1,t_1) & \cdots & K(t_1,t_q) \\ \vdots & & \vdots \\ K(t_q,t_1) & \cdots & K(t_q,t_q) \end{vmatrix} dt_1 \cdots dt_q +$$

$$+ \cdot \cdot \cdot$$

This power series is denoted by $D(\lambda)$,

(1-4) $$D(\lambda) = \lim_{n\to\infty} \Delta ,$$

and is called Fredholm's determinant. A useful notation for $D(\lambda)$ is

(1-5) $$D(\lambda) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{\lambda^n}{n!} A_n,$$

where

$$A_n = \int_a^b \cdots \int_a^b \begin{vmatrix} K(t_1,t_1) & \cdots & K(t_1,t_n) \\ \vdots & & \vdots \\ K(t_n,t_1) & \cdots & K(t_n,t_n) \end{vmatrix} dt_1 \cdots dt_n .$$

ii). Limit of Δ_{ii}. Expanding Δ_{kk} as above, we obtain the same expression as for Δ, with the exception that in the summation, $i = k$ is omitted. Thus $\lim_{n\to\infty} \Delta_{kk} = D(\lambda)$.

iii). Limit of Δ_{ij}. Expanding Δ_{ij} in the same manner we get

$$\Delta_{ij} = \lambda(\Delta t)\left[K(t_j,t_i) - \lambda\sum_{k=1}^{n}\begin{vmatrix} K(t_j,t_i) & K(t_j,t_k) \\ K(t_k,t_i) & K(t_k,t_k)\end{vmatrix}\Delta t + \right.$$

$$\left. + \frac{\lambda^2}{2!}\sum_{h,k=1}^{n}\begin{vmatrix} K(t_j,t_i) & K(t_j,t_k) & K(t_j,t_h) \\ K(t_k,t_i) & K(t_k,t_k) & K(t_k,t_h) \\ K(t_h,t_i) & K(t_h,t_k) & K(t_h,t_h)\end{vmatrix}(\Delta t)^2 + \cdots\right]$$

If we put $(\Delta t)\mathcal{D}_{ji} = \Delta_{ij}$, and let (t_j,t_i) vary in such a way that $\lim_{n\to\infty}(t_j,t_i) = (x,y)$, we get

$$\lim_{n\to\infty}\mathcal{D}_{ji} = \lambda K(x,y) - \lambda^2\int_a^b\begin{vmatrix} K(x,y) & K(x,t) \\ K(t,y) & K(t,t)\end{vmatrix}dt +$$

$$\frac{\lambda^3}{2!}\int_a^b\int_a^b\begin{vmatrix} K(x,y) & K(x,t_1) & K(x,t_2) \\ K(t_1,y) & K(t_1,t_1) & K(t_1,t_2) \\ K(t_2,y) & K(t_2,t_1) & K(t_2,t_2)\end{vmatrix}dt_1dt_2 + \cdots$$

This limit defines for us Fredholm's first minor $D(x,y;\lambda)$, that is,

$$\lim_{n\to\infty} \mathcal{D}_{ji} = D(x,y;\lambda). \tag{1-6}$$

An alternate notation for $D(x,y;\lambda)$ is

$$D(x,y;\lambda) = \lambda K(x,y) + \sum_{n=1}^{\infty} (-1)^n \frac{\lambda^{n+1}}{n!} B_n(x,y), \tag{1-7}$$

where

$$B_n(x,y) = \int_a^b \cdots \int_a^b \begin{vmatrix} K(x,y) & \cdots & K(x,t_n) \\ K(t_1,y) & \cdots & K(t_1,t_n) \\ \vdots & & \vdots \\ K(t_n,y) & \cdots & K(t_n,t_n) \end{vmatrix} dt_1 \cdots dt_n,$$

or $D(x,y;\lambda) = \sum_{n=0}^{\infty} (-1) \frac{\lambda^{n+1}}{n!} B_n(x,y)$, where $B_o(x,y) = K(x,y)$ and $0! = 1$.

We now make use of the definitions (1-4) and (1-6) and the following theorem to obtain two fundamental relations between $D(\lambda)$ and $D(x,y;\lambda)$.

Theorem: The sum of the products of the elements of any row (column) of a determinant by the corresponding minors of any other row (column) is zero.

Applying this to the j-th row of the determinant Δ we obtain

$$[1 - \lambda K(t_j,t_j)\Delta t]\Delta_{kj} - \lambda K(t_j,t_k)\Delta t(\Delta_{kk})$$

$$- \sum_{i=1}^{n} \lambda K(t_j,t_i)\Delta t(\Delta_{ki}) = 0,$$

where in the summation we omit $i = j,k$. But since $\Delta_{ki} = \Delta t \mathscr{D}_{ik}$, this becomes

$$[1 - \lambda K(t_j,t_j)\Delta t]\Delta t \mathscr{D}_{jk} - \lambda K(t_j,t_k)(\Delta_{kk})(\Delta t) -$$

$$- \sum_{i=1}^{n} \lambda K(t_j,t_i)(\Delta t)^2 \mathscr{D}_{ik} = 0.$$

Now dividing by Δt and taking the limit as n increases indefinitely we have

$$D(t_j,t_k;\lambda) - \lambda K(t_j,t_k)D(\lambda) - \lambda\int_a^b D(t,t_k;\lambda)K(t_j,t)dt = 0.$$

Since this holds for all x in [a,b], we can let $t_j = x$, $t_k = y$, so that

(1-8) $$D(x,y;\lambda) - \lambda K(x,y)D(\lambda) = \lambda\int_a^b D(t,y;\lambda)K(x,t)dt.$$

Applying this theorem to the j-th column we obtain

$$[1 - \lambda K(t_j,t_j)\Delta t]\Delta_{jk} - \lambda K(t_k,t_j)\Delta t(\Delta_{kk}) -$$

$$- \sum_{i=1}^{n} \lambda K(t_i,t_j)\Delta t(\Delta_{ik}) = 0,$$

where in the summation we omit $i = j,k$. Again, making use of $\Delta t \mathscr{D}_{ij} = \Delta_{ji}$, dividing by Δt, taking the limit as n increases indefinitely, and replacing t_k by x and t_j by y, we have

(1-9) $$D(x,y;\lambda) - \lambda K(x,y)D(\lambda) = \lambda\int_a^b D(x,t;\lambda)K(t,y)dt.$$

Equations (1-8) and (1-9) are known as Fredholm's Fundamental relations. They have been derived heuristically. In order to verify that these two relations are indeed true, we substitute in them the series expressions for $D(\lambda)$, $D(x,y;\lambda)$ as given by (1-5) and (1-7). This will make both sides of (1-8) and (1-9) power series in λ. Then if it can be shown that the coefficients of corresponding powers of λ are equal, the relations will be established.

Substituting in (1-9) the series expressions for $D(\lambda)$, $D(x,y;\lambda)$, we obtain

$$\lambda K(x,y) + \sum_{n=1}^{\infty} (-1)^n \frac{\lambda^{n+1}}{n!} B_n(x,y) - \lambda K(x,y) \left[1 + \sum_{n=1}^{\infty} (-1)^n \frac{\lambda^n}{n!} A_n \right] =$$

$$\lambda \int_a^b K(t,y) \left[\lambda K(x,t) + \sum_{n=1}^{\infty} (-1)^n \frac{\lambda^{n+1}}{n!} B_n(x,t) \right] dt.$$

On the right hand side make the following change in notation: let $n' = n+1$, and then drop the prime. Then interchanging the order of integration and summation on the right we have

$$\sum_{n=1}^{\infty} (-1)^n \frac{\lambda^{n+1}}{n!} B_n(x,y) - K(x,y) \sum_{n=1}^{\infty} (-1)^n \frac{\lambda^{n+1}}{n!} A_n =$$

$$\lambda^2 \int_a^b K(t,y) K(x,t)\,dt + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{\lambda^{n+1}}{(n-1)!} \int_a^b B_{n-1}(x,t) K(t,y)\,dt.$$

Therefore, comparing coefficients of corresponding powers of λ we see that in order for (1-9) to hold, it must be that

$$(-1)^n \frac{1}{n!} B_n(x,y) - (-1)^n \frac{K(x,y)}{n!} A_n = \frac{(-1)^{n-1}}{(n-1)!} \int_a^b B_{n-1}(x,t) K(t,y)\,dt,$$

or

$$(1\text{-}10) \qquad B_n(x,y) = K(x,y)A_n - n\int_a^b B_{n-1}(x,t)K(t,y)\,dt$$

Treating (1-12) the same way we arrive at the condition

$$(1\text{-}11) \qquad B_n(x,y) = K(x,y)A_n - n\int_a^b K(x,t)B_{n-1}(t,y)\,dt$$

Thus if (1-10) and (1-11) can be shown to be true, the two relations (1-8) and (1-9) will be established. This is done by expanding the determinant in the explicit expression for $B_n(x,y)$ and by making a suitable change in notation. From (1-7) we have that

$$B_n(x,y) = \int_a^b \cdots \int_a^b \begin{vmatrix} K(x,y) & K(x,t_1) & \cdots & K(x,t_n) \\ K(t_1,y) & K(t_1,t_1) & \cdots & K(t_1,t_n) \\ \vdots & \vdots & & \vdots \\ K(t_n,y) & K(t_n,t_1) & \cdots & K(t_n,t_n) \end{vmatrix} dt_1 \cdots dt_n$$

Now expand the determinant by elements of the first column. This yields

$$B_n(x,y) = \int_a^b \cdots \int_a^b K(x,y) \begin{vmatrix} K(t_1,t_1) & \cdots & K(t_1,t_n) \\ \vdots & & \vdots \\ K(t_n,t_1) & \cdots & K(t_n,t_n) \end{vmatrix} dt_1 \cdots dt_n +$$

$$(1\text{-}12) \qquad \sum_{i=1}^{n} (-1)^i \int_a^b \cdots \int_a^b K(t_i,y) \begin{vmatrix} K(x,t_1) & \cdots & K(x,t_n) \\ \vdots & & \vdots \\ K(t_{i-1},t_1) & \cdots & K(t_{i-1},t_n) \\ K(t_{i+1},t_1) & \cdots & K(t_{i+1},t_n) \\ \vdots & & \vdots \\ K(t_n,t_1) & \cdots & K(t_n,t_n) \end{vmatrix} dt_1 \cdots dt_n$$

In the first term the factor $K(x,y)$ can be taken outside the n-fold integral, and the first term is $K(x,y)A_n$, by the definition of A_n in (1-5). In the summation we make the change of notation $t_i = t$, $t_{i+1} = t_i$, $t_{i+2} = t_{i+1}$, ... Hence the determinant becomes

$$\begin{vmatrix} K(x,t_1) & \cdots & K(x,t_{i-1}) & K(x,t) & K(x,t_{i+1}) & \cdots & K(x,t_{n-1}) \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ K(t_{n-1},t_1) & \cdots & K(t_{n-1},t_{i-1}) & K(t_{n-1},t) & K(t_{n-1},t_{i+1}) & \cdots & K(t_{n-1},t_{n-1}) \end{vmatrix}$$

The column involving t can be brought into the first column by i-1 moves, that is by i-1 changes of sign. Therefore the second term in (1-12) can be written as

$$\sum_{i=1}^{n} (-1)^{2i-1} \int_a^b \cdots \int_a^b K(t,y) \begin{vmatrix} K(x,t) & \cdots & K(x,t_{n-1}) \\ K(t_1,t) & \cdots & K(t_1,t_{n-1}) \\ \vdots & & \vdots \\ K(t_{n-1},t) & \cdots & K(t_{n-1},t_{n-1}) \end{vmatrix} dt \ldots dt_n$$

In this sum, all n terms will be equal, and we may bring the $K(t,y)$ factor outside the n-1 fold integral, so that the second term in (1-12) reduces to

$$-n \int_a^b K(t,y) \left[\int^b \cdots \int^b \begin{vmatrix} K(x,t) & \cdots & K(x,t_{n-1}) \\ \vdots & & \vdots \\ K(t_{n-1},t) & \cdots & K(t_{n-1},t_{n-1}) \end{vmatrix} dt_1 \ldots dt_{n-1} \right] dt ,$$

which from (1-7) is $-n \int_a^b K(t,y) B_{n-1}(x,t)dt$. Hence

$$B_n(x,y) = K(x,y)A_n - n \int_a^b K(t,y)B_{n-1}(x,t)dt ,$$

which establishes (1-10) .

If, now , we take the expression for $B_n(x,y)$ and expand it by elements of the first row, make the same change in notation , we get (1-11) .

Thus we have established that between Fredholm's determinant and first minor the following double relation holds :

$$D(x,y;\lambda) - \lambda K(x,y)D(\lambda) = \lambda\int_a^b D(x,t;\lambda)K(t,y)dt \tag{1-13}$$
$$= \lambda\int_a^b D(t,y;\lambda)K(x,t)dt$$

for all λ and all x and y satisfying $a \leq x \leq b$, $a \leq y \leq b$.

1.2 SOLUTION OF FREDHOLM'S EQUATION FOR $D(\lambda) \neq 0$

We are now in a position to solve equation (1) , but to obtain a hint as to procedure , let us consider the finite system

$$u(t_i) - \lambda(\Delta t)\sum_{j=1}^{n} K(t_i,t_j)u(t_j) = f(t_i) \ , \ i = 1, \ . \ . \ . \ n$$

To solve this system for $u(t_k)$, multiply by Δ_{ik} and sum with respect to i from 1 to n , thus obtaining

$$\sum_{i=1}^{n} u(t_i)\Delta_{ik} - \lambda\Delta t\sum_{i=1}^{n}\sum_{j=1}^{n} K(t_i,t_j)u(t_j)\Delta_{ik} = \sum_{i=1}^{n} f(t_i)\Delta_{ik}$$
$$= \Delta u(t_k)$$

But $\Delta_{ik} = (\Delta t)\mathcal{D}_{ki}$, and $\lim_{n\to\infty}\sum_{i=1}^{n}\mathcal{D}_{ki}\Delta t = \int_a^b D(x,t;\lambda)dt$, from (1-10) . Let us now make use of the analogy, first rewriting (1) as

$$u(t) = f(t) + \lambda \int_a^b K(t,r)u(r)dr \tag{1-14}$$

Now multiply by $D(x,t;\lambda)$ and integrate from a to b with respect to t .

We get

$$\int_a^b D(x,t;\lambda)u(t)dt = \int_a^b D(x,t;\lambda)f(t)dt + \lambda \int_a^b \int_a^b D(x,t;\lambda)K(t,r)u(r)drdt$$

$$= \int_a^b D(x,t;\lambda)f(t)dt + \int_a^b u(r)\left[\lambda \int_a^b D(x,t;\lambda)K(t,r)dt\right]dr$$

According to (1-9) the term in brackets in the last integral becomes

$$\left[D(x,r;\lambda) - \lambda K(x,r)D(\lambda)\right] .$$

Therefore

$$\int_a^b D(x,t;\lambda)u(t)dt = \int_a^b D(x,t;\lambda)f(t)dt + \int_a^b D(x,r;\lambda)u(r)dr - \lambda \int_a^b K(x,r)D(\lambda)u(r)dr .$$

Here note that the term on the left and the second term on the right are the same except for the variable of integration . Therefore

$$0 = \int_a^b D(x,t;\lambda)f(t)dt - D(\lambda)\,\lambda \int_a^b K(x,r)u(r)dr .$$

But from (1-14) this last term is equal to $D(\lambda)\left[u(x) - f(x)\right]$. Hence

$$0 = \int_a^b D(x,t;\lambda)f(t)dt - D(\lambda)\left[u(x) - f(x)\right] ,$$

$$u(x) = f(x) + \frac{1}{D(\lambda)} \int_a^b D(x,t;\lambda)f(t)dt . \tag{1-15}$$

Equation (1-15) gives us u(x) in terms of the known function f(x) and the ratio of the two power series in λ. Thus if (1-14) has any solution it must be that given by (1-15).

To show that (1-15) is indeed a solution of (1-14), we substitute this value of u(x) into (1-14). The left side becomes

$$f(x) + \int_a^b \frac{D(x,t;\lambda)f(t)dt}{D(\lambda)}$$

and the right side becomes

$$f(x) + \lambda\int_a^b K(x,t)\left[f(t) + \int_a^b \frac{D(t,r;\lambda)f(r)dr}{D(\lambda)}\right]dt$$

$$= f(x) + \lambda\int_a^b K(x,t)f(t)dt + \frac{1}{D(\lambda)}\int_a^b f(r)\left[\lambda\int_a^b K(x,t)D(t,r;\lambda)dt\right]dr$$

$$= f(x) + \lambda\int_a^b K(x,t)f(t)dt + \frac{1}{D(\lambda)}\int_a^b f(r)\left[D(x,r;\lambda) - K(x,r)D(\lambda)\right]dr$$

$$= f(x) + \lambda\int_a^b K(x,t)f(t)dt + \frac{1}{D(\lambda)}\int_a^b f(r)D(x,r;\lambda)dr - \lambda\int_a^b K(x,r)f(r)dr$$

$$= f(x) + \frac{1}{D(\lambda)}\int_a^b f(r)D(x,r;\lambda)dr ,$$

since the second and fourth terms are the same except for the variable of integration. Thus (1-14) is satisfied by the value of u(x) given in (1-15).

It remains now to show that $D(\lambda)$ and $D(x,y;\lambda)$ are absolutely convergent power series in λ, each with an infinite radius of convergence. To do this we need a theorem due to Hadamard, which in turn requires the following:

Lemma: If all the elements of the determinant

$$A = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

are real and satisify

(1-16) $$\sum_{s=1}^{n} a_{rs}^2 = 1 \ , \ r = 1, \ldots n$$

then $|A| \leq 1$.

proof:

$A(a_{ij})$ is a continuous function of its arguments , a_{ij} , in the region defined by (1-16) , and hence somewhere in this region it reaches a maximum and a minimum . But if a function f of N variables , with the n side conditions $\phi_i = 0$, $i = 1, \ldots n$, is to have a maximum (minimum) , then the N first partial derivitives of the function

$$f + \lambda_1 \phi_1 + \ldots \lambda_n \phi_n$$

must vanish , where λ_i are the Lagrange multipliers [1] . In our case $f = A$ and $\phi_i = \sum_{s=1}^{n} a_{rs}^2 - 1$, $r = 1, \ldots n$. Therefore for a maximum (minimum) to exist the N first partial derivitives of

$$F = A + \sum_{s=1}^{n} \frac{\lambda s}{2}(a_{s1}^2 + \ldots + a_{sn}^2 - 1)$$

must vanish . But

$$A = \sum_{s=1}^{n} a_{rs} A_{rs} ,$$

[1] Kaplan , Advanced Calculus , Addison - Wesley (1952) §2-16

where r is one of the numbers $1, 2, \ldots, n$, and where A_{rs} is the cofactor of a_{rs}. Therefore

$$F = \sum_{s=1}^{n} a_{rs}A_{rs} + \sum_{s=1}^{n} \frac{\lambda r}{2}(a_{r1}^2 + \ldots a_{rn}^2 - 1) .$$

Hence for a maximum (minimum) to exist

$$\frac{\partial F}{\partial a_{rs}} = A_{rs} + \lambda_r a_{rs} = 0 . \tag{1-17}$$

Now multiply (1-17) by a_{rs} and sum with respect to s from one to n. We obtain

$$\sum_{s=1}^{n} a_{rs}A_{rs} + \sum_{s=1}^{n} \lambda_r a_{rs}^2 = 0 .$$

But this gives $A + \lambda_r = 0$. Thus for a maximum (minimum) we must have $\lambda_r = -A$. Putting this back into (1-17) we have $A_{rs} = Aa_{rs}$. Thus

$$\begin{aligned} \det(A_{rs}) &= \det(Aa_{rs}) \\ &= A^n \det(a_{rs}) \\ &= A^{n+1} . \end{aligned}$$

But

$$\begin{aligned} \det(A_{rs}) &= \frac{1}{A}\det(a_{rs})\det(A_{rs}) \\ &= \frac{1}{A}\det(a_{rk}A_{ks}) \\ &= \frac{1}{A}\det(\delta^r_s A) \\ &= A^{n-1} . \end{aligned}$$

Therefore

$$A^{n+1} = A^{n-1} \tag{1-18}$$

$$A^2 = 1 .$$

Then since at a maximum or minimum of A we must have $A^2 = 1$, the maximum of A is +1 and the minimum is -1 . Therefore $|A| \leq 1$.

We make use of this lemma to prove Hadamard's theorem: If all the elements of $\det(b_{rs})$ are real and satisfy the inequality $|b_{ij}| \leq M$, then $|B| \leq M^n\sqrt{n^n}$.

proof :

Let $\sum_{j=1}^{n} b_{ij}^2 = s_i$, $i = 1, \ldots n$

Case i) : some one of the $s_i = 0$, say it is s_j . Then $b_{ij} = 0$, $i = 1, \ldots n$, and hence $|B| = 0$ and the theorem is proved .

Case ii) : no $s_i = 0$. Since $B = \det(b_{ij})$,

$$\frac{B}{\sqrt{s_1 s_2 \ldots s_n}} = \det\left(\frac{b_{ij}}{\sqrt{s_i}}\right).$$

This determinant satisfies the conditions of our lemma , hence

$$\frac{|B|}{\sqrt{s_1 s_2 \ldots s_n}} \leq 1 .$$

Thus $|B| \leq \sqrt{s_1 s_2 \ldots s_n}$. But since $|b_{ij}| \leq M$, $s_i \leq nM^2$. Therefore

$$|B| \leq \sqrt{(nM^2)^n}$$

$$|B| \leq M^n\sqrt{n^n} .$$

With the use of Hadamard's theorem , we now show that $D(\lambda)$ and $D(x,y;\lambda)$ are absolutely convergent series in λ .

Convergence of $D(\lambda)$. From (1-5)

$$D(\lambda) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{\lambda^n}{n!} A_n ,$$

where

$$A_n = \int_a^b \cdots \int_a^b \begin{vmatrix} K(t_1,t_1) & \cdots & K(t_1,t_n) \\ \vdots & & \vdots \\ K(t_n,t_1) & \cdots & K(t_n,t_n) \end{vmatrix} dt_1 \cdots dt_n$$

The determinant in A_n satisfies the conditions of Hadamards theorem, because it was assumed originally that $K(x,t)$ is continuous in and on the boundary of the rectangle $a \leq x \leq b$, $a \leq t \leq b$. Hence $K(t_i,t_j) \leq M$, i , $j = 1, \ldots n$. Thus Hadamard's theorem gives us

$$|A_n| \leq \int_a^b \cdots \int_a^b M^n \sqrt{n^n}\, dt_1 \cdots dt_n$$

$$= M^n \sqrt{n^n}\,(b - a)^n .$$

Therefore

$$\left| (-1)^n \frac{\lambda^n}{n!} A_n \right| \leq M^n \sqrt{n^n}\,(b - a)^n \left| \frac{\lambda^n}{n!} \right| = C_n$$

Now apply the ratio test to the series of which C_n is the general term. We find that

$$\frac{C_{n+1}}{C_n} = M|\lambda|(b - a)\sqrt{\left(\frac{n+1}{n}\right)^n}\ \frac{\sqrt{n+1}}{n+1}$$

and therefore $\lim\limits_{n\to\infty} \dfrac{C_{n+1}}{C_n} = 0$, so that the series for $D(\lambda)$ converges ~~absolutely~~ .

~~Convergence of $D(x,y;\lambda)$. From (1-7)~~

$$D(x,y;\lambda) = \sum_{n=0}^{\infty} (-1)^n \frac{\lambda^{n+1}}{n!} B_n(x,y) ,$$

where

$$B_n(x,y) = \int_a^b \cdots \int_a^b \begin{vmatrix} K(x,y) & \cdots & K(x,t_n) \\ \vdots & & \vdots \\ K(t_n,y) & \cdots & K(t_n,t_n) \end{vmatrix} dt_1 \cdots dt_n$$

The determinant in $B_n(x,y)$ also satisfies the conditions of Hadamard's theorem , hence

$$|B| \leq \int_a^b \cdots \int_a^b \sqrt{(n+1)^{n+1}}\, M^{n+1}\, dt_1 \cdots dt_n$$

$$= M^{n+1} \sqrt{(n+1)^{n+1}}\, (b - a)^n .$$

Therefore

$$\left| (-1)^n \frac{\lambda^{n+1}}{n!} B_n \right| \leq M^{n+1} \sqrt{(n+1)^{n+1}}\, (b - a)^n \frac{|\lambda^{n+1}|}{n!} = E_n .$$

Again applying the ratio test to the series of which E_n is the general term we have

$$\frac{E_{n+1}}{E_n} = M|\lambda|(b - a) \sqrt{\left(\frac{n+2}{n+1}\right)^{n+1}} \sqrt{\frac{n+2}{n}} .$$

Therefore $\lim\limits_{n\to\infty} \frac{E_{n+1}}{E_n} = 0$, so that the series for $D(x,y;\lambda)$ is also absolutely convergent .

As a direct consequence of the convergence proofs we note that

$$\lim_{n\to\infty} \frac{C_n}{C_{n+1}} = \infty , \lim_{n\to\infty} \frac{E_n}{E_{n+1}} = \infty :$$

But these limits are by definition the radii of convergence of the power series, so that both series converge for all values of λ.

We have thus proved Fredholm's first fundamental theorem :

Theorem 1-1 : If

i) $D(\lambda) \neq 0$

ii) $K(x,y)$ is continuous, real, and not identically zero in the rectangle $a \leq x \leq b$, $a \leq y \leq b$,

iii) $f(x)$ is continuous, real, and not identically zero in the interval $a \leq x \leq b$,

then the equation

$$(1) \qquad u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt$$

has one and only one continuous solution given by

$$(1\text{-}15) \qquad u(x) = f(x) + \frac{1}{D(\lambda)} \int_a^b D(x,t;\lambda)f(t)dt ,$$

where $D(\lambda)$ and $D(x,t;\lambda)$ are absolutely convergent series in λ, each with an infinite radius of convergence.

As an immediate corollary to Theorem 1-1 we have :

Corollary 1-1 : If $D(\lambda) \neq 0$, then the homogeneous equation with $f(x) \equiv 0$ has one and only one continuous solution, and that is the trivial solution $u(x) \equiv 0$.

1.3 SOLUTION OF THE HOMOGENEOUS EQUATION FOR $D(\lambda) = 0$

Obviously the method of the previous section for solving equation (1) fails if $D(\lambda) = 0$. We now formulate a method of obtaining the solution when $D(\lambda) = 0$, first with respect to the homogenous equation

$$u(x) = \lambda \int_a^b K(x,t)u(t)dt \quad . \tag{1-19}$$

Case i) : $D(x,y;\lambda) \not\equiv 0$

Suppose there exists a λ_0 such that $D(\lambda_0) = 0$. Then using this value of λ , equation (1-8) becomes

$$D(x,y;\lambda_0) = \lambda_0 \int_a^b D(t,y;\lambda_0)K(x,t)dt \tag{1-20}$$

Since (1-20) holds for all y in $[a,b]$, we can choose $y = y_0$. Then

$$D(x,y_0;\lambda_0) = \lambda_0 \int_a^b K(x,t)D(t,y_0;\lambda_0)dt \tag{1-21}$$

Note that (1-21) is the same as (1-19) except that $u(x)$ has been replaced by $D(x,y_0;\lambda_0)$. Thus $D(x,y_0;\lambda_0)$ is a solution of (1-19) . Should $D(x,y_0;\lambda_0) = 0$, we could choose a different value of y , but if $D(x,y;\lambda_0) \equiv 0$, then the above method gives only the trivial solution $u(x) \equiv 0$.

Thus if $D(x,y;\lambda_0) \not\equiv 0$, then for a suitable choice of y , $D(x,y_0;\lambda_0)$ is a nontrivial solution of (1-19) .

The condition $D(x,y;\lambda_0) \not\equiv 0$ can be replaced by $D'(\lambda) \neq 0$, for consider the series for $D(\lambda)$:

$$D(\lambda) = 1 - \lambda A_1 + \frac{\lambda^2}{2!}A_2 - \frac{\lambda^3}{3!}A_3 + \dots .$$

Hence

$$D'(\lambda) = -A_1 + \lambda A_2 - \frac{\lambda^2}{2!}A_3 + \dots = -\sum_{n=0}^{\infty} (-1)^n \frac{\lambda^n}{n!} A_{n+1} \quad .$$

But

$$A_{n+1} = \int_a^b \cdots \int_a^b \begin{vmatrix} K(t_1,t_1) & \cdots & K(t_1,t_{n+1}) \\ \vdots & & \vdots \\ K(t_{n+1},t_1) & \cdots & K(t_{n+1},t_{n+1}) \end{vmatrix} dt_1 \cdots dt_{n+1} ,$$

and if we make the change in notation $x = t_1$, $t_1 = t_2$, . . . , $t_n = t_{n+1}$, this becomes

$$A_n = \int_a^b \left[\int_a^b \cdots \int_a^b \begin{vmatrix} K(x,x) & \cdots & K(x,t_n) \\ \vdots & & \vdots \\ K(t_n,x) & \cdots & K(t_n,t_n) \end{vmatrix} dt_1 \cdots dt_n \right] dx$$

$$= \int_a^b B_n(x,x)dx ,$$

by the definition (1-7) . Therefore

$$D'(\lambda) = -\sum_{n=0}^{\infty} (-1)^n \frac{\lambda^n}{n!} \int_a^b B_n(x,x)\, dx$$

$$= -\int_a^b \sum_{n=0}^{\infty} (-1)^n \frac{\lambda^n}{n!} B_n(x,x)dx$$

$$= -\frac{1}{\lambda} \int_a^b D(x,x;\lambda)dx ,$$

from (1-7) , so that

(1-22) $$-\lambda D'(\lambda) = \int_a^b D(x,x;\lambda)dx .$$

Now suppose $D(\lambda_o) = 0$, $D'(\lambda_o) \neq 0$; then $\lambda_o \neq 0$. Thus from (1-22) if $D'(\lambda_o) \neq 0$ and $\lambda_o \neq 0$, $D(x,x;\lambda_o) \neq 0$ in x , and consequently $D(x,y;\lambda_o) \neq 0$ in x and y . Therefore the condition $D(x,y;\lambda_o) \neq 0$ is equivalent to $D'(\lambda_o) \neq 0$.

The relation (1-22) between the first derivitive and the first minor will later be generalized to the p-th derivitive and the p-th minor .

Case ii) : $D(x,y;\lambda_o) \equiv 0$.

In order to get a solution other than the trivial one , we must consider minors of Δ of higher orders . Define the p-th minor of $D(\lambda)$ by

$$(1\text{-}23) \qquad D\begin{pmatrix} x_1 \cdots x_p \\ y_1 \cdots y_p \end{pmatrix} \lambda = \sum_{n=0}^{\infty} (-1)^n \frac{\lambda^{n+p}}{n!} B_n\begin{pmatrix} x_1 \cdots x_p \\ y_1 \cdots y_p \end{pmatrix}$$

$$\equiv D_p(x,y;\lambda_o)$$

where

$$B_n\begin{pmatrix} x_1 \ldots x_p \\ y_1 \ldots y_p \end{pmatrix} = \int_a^b \cdots \int_a^b \begin{vmatrix} K(x_1,y_1)\ldots K(x_1,y_p) & K(x_1,t_1)\ldots K(x_1,t_n) \\ \vdots \qquad\qquad \vdots & \vdots \qquad\qquad \vdots \\ K(x_p,y_1)\ldots K(x_p,y_p) & K(x_p,t_1)\ldots K(x_p,t_n) \\ K(t_1,y_1)\ldots K(t_1,y_p) & K(t_1,t_1)\ldots K(t_1,t_n) \\ \vdots \qquad\qquad \vdots & \vdots \qquad\qquad \vdots \\ K(t_n,y_1)\ldots K(t_n,y_p) & K(t_n,t_1)\ldots K(t_n,t_n) \end{vmatrix} dt_1 \ldots dt_n$$

or

$$(1\text{-}24)\quad B_n\begin{pmatrix} x_1 \cdots x_p \\ y_1 \cdots y_p \end{pmatrix} = \int_a^b \cdots \int_a^b K\begin{pmatrix} x_1 \cdots x_p, t_1 \cdots t_n \\ y_1 \cdots y_p, t_1 \cdots t_n \end{pmatrix} dt_1 \cdots dt_n \ .$$

We now show that the series for $D_p(x,y;\lambda_o)$ is absolutely convergent in λ . To do this we may use Hadamard's theorem since the determinant in the series satisfies the necessary conditions . Therefore

$$|B_n| \le \int_a^b \cdots \int_a^b \sqrt{(n+p)^{n+p}}\, M^{n+p}\, dt_1 \cdots dt_n \ ,$$

where $|K(x_\alpha, y_\beta)| \le M$, for $\alpha, \beta = 1, \ldots p$. Hence

$$|B_n| \le \sqrt{(n+p)^{n+p}}\, M^{n+p} (b - a)^n \ ,$$

$$\left| (-1)^n \frac{\lambda^{n+p}}{n!} B_n \right| \le \frac{\lambda^{n+p}}{n!} \sqrt{(n+p)^{n+p}}\, M^{n+p} (b - a)^n = F_n$$

The series of which F_n is the general term is absolutely convergent , for using the ratio test we have

$$\frac{F_n}{F_{n+1}} = \frac{\lambda}{n} M(b - a) \sqrt{\left(\frac{n+p}{n+p-1}\right)^{n+p-1}} \sqrt{(n+p)} \ .$$

Therefore $\lim\limits_{n\to\infty} \dfrac{F_n}{F_{n-1}} = 0$, and the series converges .

Before we can obtain a solution of (1-19) , it becomes necessary to generalize some previous results . We will extend the fundamental relations (1-8) and (1-9) , as well as (1-22) to the p-th minor and $D^{(p)}(\lambda)$.

To obtain Fredholm's generalized fundamental relations , we expand the determinant in $B_n\begin{pmatrix} x_1, \dots x_p \\ y_1, \dots y_p \end{pmatrix}$, first by elements of the column y_β, and then by elements of the row x_α. By making a suitable change in notation, and multiplying by $\frac{\lambda^{n+p}}{n!}$ and summing with respect to n from zero to infinity, we arrive at the following double relation :

$$D_p(x,y;\lambda) = \sum_{\alpha=1}^{p} (-1)^{\alpha+\beta} \lambda K(x_\alpha, y_\beta) D\begin{pmatrix} x_1,\dots,x_{\alpha-1}, x_{\alpha+1},\dots x_p \\ y_1,\dots,y_{\beta-1}, y_{\beta+1},\dots y_p \end{pmatrix}\lambda + \tag{1-25}$$

$$\lambda \int_a^b K(t,y_\beta) D\begin{pmatrix} x_1,\dots,x_{\beta-1}, x_\beta, x_{\beta+1},\dots,x_p \\ y_1,\dots,y_{\beta-1}, t, y_{\beta+1},\dots,y_p \end{pmatrix}\lambda \, dt$$

$$D_p(x,y;\lambda) = \sum_{\beta=1}^{p} (-1)^{\alpha+\beta} \lambda K(x_\alpha, y_\beta) D\begin{pmatrix} x_1,\dots,x_{\alpha-1}, x_{\alpha+1},\dots,x_p \\ y_1,\dots,y_{\beta-1}, y_{\beta+1},\dots,y_p \end{pmatrix}\lambda + \tag{1-26}$$

$$\lambda \int_a^b K(x_\alpha,t) D\begin{pmatrix} x_1,\dots,x_{\alpha-1}, t, x_{\alpha+1},\dots,x_p \\ y_1,\dots,y_{\alpha-1}, y_\alpha, y_{\alpha+1},\dots,y_p \end{pmatrix}\lambda \, dt \; .$$

Now consider the series for $D(\lambda)$.

$$D(\lambda) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{\lambda^n}{n!} A_n \; .$$

Therefore

$$D^{(p)}(\lambda) = \sum_{n=p}^{\infty} (-1)^n \frac{\lambda^{n-p}}{(n-p)!} A_n$$

$$D^{(p)}(\lambda) = \sum_{n=0}^{\infty} (-1)^{n+p} \frac{\lambda^n}{n!} A_{n+p} \tag{1-27}$$

But

$$A_{n+p} = \int_a^b \cdots \int_a^b K\begin{pmatrix} t_1, \ldots t_{n+p} \\ t_1, \ldots t_{n+p} \end{pmatrix} dt_1 \ldots dt_{n+p}$$

Now make the following change in notation: let $t_1 = x_1, \ldots,$ $t_p = x_p, t_{p+1} = t_1, \ldots, t_{p+n} = t_n$. Thus

$$\text{(1-28)} \quad A_{n+p} = \int_a^b \cdots \int_a^b \left[\int_a^b \cdots \int_a^b K\begin{pmatrix} x_1 \ldots x_p, t_1 \ldots t_n \\ x_1 \ldots x_p, t_1 \ldots t_n \end{pmatrix} dt_1 \ldots dt_n \right] dx_1 \ldots dx_p$$

$$= \int_a^b \cdots \int_a^b B_n \begin{pmatrix} x_1 \ldots x_p \\ x_1 \ldots x_p \end{pmatrix} dx_1 \ldots dx_p$$

Multiply (1-28) by $(-1)^n \dfrac{\lambda^{n+p}}{n!}$ and sum from $n = 0$ to $n = \infty$. We thus have

$$\sum_{n=0}^{\infty} (-1)^n \frac{\lambda^{n+p}}{n!} A_{n+p} = \sum_{n=0}^{\infty} \int_a^b \cdots \int_a^b (-1)^n \frac{\lambda^{n+p}}{n!} B_n \begin{pmatrix} x_1 \ldots x_p \\ x_1 \ldots x_p \end{pmatrix} dx_1 \ldots dx_p$$

But from (1-27) and (1-23) this is

$$\text{(1-29)} \quad (-1)^p \lambda^p D^{(p)}(\lambda) = \int_a^b \cdots \int_a^b D\begin{pmatrix} x_1 \ldots x_p \\ x_1 \ldots x_p \end{pmatrix} \lambda \Bigg) dx_1 \ldots dx_p .$$

This generalizes (1-22) to the p-th derivitive and p-th minor.

We will use equation (1-29) to show that if $D(\lambda) \neq 0$, not all the Fredholm minors are zero. Suppose that $D(\lambda) \neq 0$, but that for λ_0 $D(\lambda_0) = 0$, $D'(\lambda_0) = 0$, $D''(\lambda_0) = 0, \ldots, D^{r-1}(\lambda_0) = 0$, but $D^r(\lambda_o) \neq 0$. Then λ_o is a root of $D(\lambda) = 0$ of finite multiplicity r. Note that the multiplicity must be finite, for if not $D(\lambda) \equiv$ constant, and since for some value of λ, namely $\lambda = \lambda_o$, the constant is zero, we would have $D(\lambda) \equiv 0$, which contradicts our assumption. Thus in the expression for $D(\lambda)$ we must come to some derivitive which is not zero.

Now apply this to (1-29). Suppose that for $\lambda = \lambda_o$, $D(\lambda_o) = 0$, $D^i(\lambda_o) = 0$, $i = 1, \ldots p-1$, but that $D^p(\lambda_o) \neq 0$. Note that $\lambda_o \neq 0$, for $D(0) = 1$. Therefore the left side of (1-29) is not equal to zero, hence the right side is not identically zero in x, that is $D\begin{pmatrix} x_1 \ldots x_p \\ x_1 \ldots x_p \end{pmatrix}\lambda_o \not\equiv 0$ in x, so that $D\begin{pmatrix} x_1 \ldots x_p \\ y_1 \ldots y_p \end{pmatrix}\lambda_o \not\equiv 0$ in x and y. This tells us that we must come to a number $q \leq r$, called the index of λ_o, such that $D_i(x,y;\lambda_o) \equiv 0$, $i = 1, \ldots q-1$, but that $D_q(x,y;\lambda_o) \neq 0$, that is there exists x_i', y_i', $i = 1, \ldots q$, such that $D_q(x',y';\lambda_o) \neq 0$.

We now make use of these results to obtain a solution of (1-19) when $D(x,y;\lambda_o) \equiv 0$. Let λ_o be a root of $D(\lambda) = 0$ of index q, and in (1-26) let $x_i = x_i'$, $i = 1, \ldots q$, $i \neq \alpha$, and $x_\alpha = x$, $y_i = y_i'$, $i = 1, \ldots, q$. Then we get

$$D\begin{pmatrix} x_1' \ldots x_{\alpha-1}', x, x_{\alpha+1}' \ldots x_q' \\ y_1' \ldots\ldots y_\beta', \ldots\ldots y_q' \end{pmatrix}\lambda_o = \sum_{\beta=1}^{q} (-1)^{\alpha+\beta} \lambda K(x,y_\beta') D\begin{pmatrix} x_1' \ldots x_{\alpha-1}', x_{\alpha+1}' \ldots x_q' \\ y_1' \ldots y_{\beta-1}', y_{\beta+1}' \ldots y_q' \end{pmatrix}$$

$$+ \lambda_o \int_a^b K(x,t) D\begin{pmatrix} x_1' \ldots x_{\alpha-1}', t, x_{\alpha+1}' \ldots x_q' \\ y_1' \ldots\ldots y_\alpha', \ldots\ldots y_q' \end{pmatrix} dt$$

But by hypothesis the first term on the right is zero, , since $D_{q-1}(x,y;\lambda_o) = 0$. Thus

(1-30) $$D\begin{pmatrix} x_1' \cdots x_{\alpha-1}', x, x_{\alpha+1}' \cdots x_q' \\ y_1' \cdots y_{\beta-1}', y_\beta', y_{\beta+1}' \cdots y_q' \end{pmatrix}\lambda_o\Bigg) = \lambda_o \int_a^b K(x,t) D\begin{pmatrix} x_1' \cdots x_{\alpha-1}', t, x_{\alpha+1}' \cdots x_q' \\ y_1' \cdots\cdots y_\alpha', \cdots\cdots y_q' \end{pmatrix} dt$$

Now let

(1-31) $$\phi_\alpha(x,\lambda_o) = \frac{D\begin{pmatrix} x_1' \cdots x_{\alpha-1}', x, x_{\alpha+1}' \cdots x_q' \\ y_1' \cdots\cdots y_\beta', \cdots\cdots y_q' \end{pmatrix}\lambda_o\Bigg)}{D\begin{pmatrix} x_1' \cdots x_q' \\ y_1' \cdots y_q' \end{pmatrix}\lambda_o\Bigg)}, \quad \alpha = 1,\ldots,q$$

and divide (1-30) by $D\begin{pmatrix} x_1' \cdots x_q' \\ y_1' \cdots y_q' \end{pmatrix}\lambda_o\Bigg)$. Hence (1-30) becomes

(1-32) $$\phi_\alpha(x,\lambda_o) = \lambda_o \int_a^b K(x,t)\,\phi_\alpha(t,\lambda_o)\,dt$$

Note that (1-32) is the same as (1-19) if $u(x)$ is replaced by $\phi_\alpha(x,\lambda_o)$. Thus the functions $\phi_\alpha(x,\lambda_o)$ are solutions of the homogeneous equation (1-19). Note that

(1-33) $$\phi_\alpha(x_\beta',\lambda_o) = \begin{cases} 1, & \text{if } \alpha = \beta \\ 0, & \text{if } \alpha \neq \beta \end{cases},$$

for if $\alpha = \beta$, (1-31) becomes

$$\phi_\alpha(x_\alpha,\lambda_o) = \frac{D\begin{pmatrix} x_1' \cdots x_{\alpha-1}', x', x_{\alpha+1}' \cdots x_q' \\ y_1' \cdots\cdots y_\alpha', \cdots\cdots y_q' \end{pmatrix}\lambda_o\Bigg)}{D\begin{pmatrix} x_1' \cdots x_q' \\ y_1' \cdots y_q' \end{pmatrix}\lambda_o\Bigg)}.$$

Therefore $\phi_\alpha(x_\alpha, \lambda_0) = 1$. But if $\alpha \neq \beta$, then $\beta = \alpha \pm n$, where n is an integer, and $\beta \leq q$, because α and β are indicies, both of which are less than or equal to q. Thus in the determinant in (1-24) two rows would be equal, hence the determinant would be zero, Therefore the numerator of (1-31) would be zero for $\alpha \neq \beta$, and $\phi_\alpha(x, \lambda_0) = 0$. The functions ϕ_i, $i = 1, \ldots, q$, are linearly independent, for suppose we have that

$$c_1\phi_1 + c_2\phi_2 + \ldots + c_q\phi_q \equiv 0,$$

where the c_i are constants. Then for $x = x'_\alpha$, $\phi_\alpha(x_\alpha, \lambda_0) = 1$ and $\phi_\alpha(x_\beta, \lambda_0) = 0$, so that $c_\alpha = 0$. Hence the functions ϕ_i are linearly independent.

From the homogeneity and linearity of equation (1-19) we have that

$$\Phi(x) = c_1\phi_1(x) + c_2\phi_2(x) + \ldots + c_q\phi_q(x) \tag{1-34}$$

is again a solution, for arbitrary constants c_i.

We now show that every solution of (1-19) can be put in the form (1-34). Suppose that v(x) is a solution of (1-19). Then

$$\begin{aligned} v(x) &= \lambda_0\int_a^b K(x,t)v(t)dt \\ 0 &= v(x) - \lambda_0\int_a^b K(x,t)v(t)dt \end{aligned} \tag{1-35}$$

Now consider the first of Fredholm's generalized relations, written with $p = q + 1$, $x_{q+1} = x$, $y_{q+1} = y$. Then (1-25) becomes

$$D\begin{pmatrix} x, x_1 \ldots x_q \\ y, y_1 \ldots y_q \end{pmatrix} \lambda_o = \lambda_o K(x,t) D\begin{pmatrix} x_1 \ldots x_q \\ y_1 \ldots y_q \end{pmatrix} \lambda_o -$$

$$\sum_{\alpha=1}^{q} \lambda_o K(x_\alpha, y) D\begin{pmatrix} x_1 \ldots x_{\alpha-1}, x, x_{\alpha+1} \ldots x_q \\ y_1 \ldots y_{\beta-1}, y_\beta, y_{\beta+1} \ldots y_q \end{pmatrix} \lambda_o +$$

$$\lambda_o \int_a^b K(t,y) D\begin{pmatrix} x, x_1 \ldots x_q \\ y, y_1 \ldots y_q \end{pmatrix} \lambda_o \, dt$$

If we now let $x_1 = x_1'$, . . . , $x_q = x_q'$, $y_1 = y_1'$, . . . , $y_q = y_q'$, $y = t$, divide by $D_q(x', y'; \lambda_o)$ and let

$$H(x,y) = \frac{D\begin{pmatrix} x, x_1' \ldots x_q' \\ y, y_1' \ldots y_q' \end{pmatrix} \lambda_o}{D\begin{pmatrix} x_1' \ldots x_q' \\ y_1' \ldots y_q' \end{pmatrix} \lambda_o} ,$$

this becomes

(1-36) $$H(x,t) = \lambda_o K(x,t) - \sum_{\alpha=1}^{q} \lambda_o K(x_\alpha', t)\, \phi_\alpha(x) + \lambda_o \int_a^b H(x,s) K(s,t) ds ,$$

where $\phi_\alpha(x)$ is defined by (1-31). If we now multiply the second equation of (1-35) by $H(x,t)$ and integrate from a to b with respect to t we have

(1-37) $$0 = \int_a^b H(x,t) \left[v(t) - \lambda_o \int_a^b K(t,s) v(s) ds \right] dt .$$

Now subtracting (1-37) from the first equation of (1-35) we get

$$(1\text{-}38)\quad v(x) = \int_a^b \left[\lambda_0 K(x,t) - H(x,t) + \lambda_0 \int_a^b H(x,s)K(s,t)ds\right] v(t)dt$$

But from (1-36) this is

$$v(x) = \int_a^b \left[\sum_{\alpha=1}^{q} \lambda_0 K(x'_\alpha,t)\,\phi_\alpha(x)\right] v(t)dt$$

$$(1\text{-}39)\qquad v(x) = \lambda_0 \sum_{\alpha=1}^{q} \phi_\alpha(x) \int_a^b K(x'_\alpha,t)v(t)dt \quad .$$

Thus if we take for the constants

$$c_\alpha = \lambda_0 \int_a^b K(x'_\alpha,t)v(t)dt \quad ,$$

(1-39) can be written in the form (1-34). Thus we arrive at Fredholm's second fundamental theorem:

Theorem 1-2: If $\lambda = \lambda_0$ is a root of $D(\lambda) = 0$ of index q, then the homogeneous equation (1-19) has q linearly independent solutions given by

$$\phi_\alpha(x) = \frac{D\begin{pmatrix} x'_1 \ldots x'_{\alpha-1},\, x, x'_{\alpha+1} \ldots x'_q & \\ y'_1 \ldots y'_{\beta-1}, y'_\beta, y'_{\beta+1} \ldots y'_q & \lambda_0 \end{pmatrix}}{D\begin{pmatrix} x'_1 \ldots x'_q & \\ y'_1 \ldots y'_q & \lambda_0 \end{pmatrix}}$$

in terms of which every other solution is expressible linearly and homogeneously.

The parameter λ_0 such that $D(\lambda_0) = 0$ is called the characteristic constant of the kernel $K(x,t)$, and the function $\phi(x)$ which satisfies (1-19) is called the characteristic function belonging to λ_0.

1.4 SOLUTION OF THE NON-HOMOGENEOUS EQUATION FOR $D(\lambda) = 0$.

As in the case of the finite system of linear equations for which the determinant of the coefficients is zero, the non-homogeneous linear integral equation with $D(\lambda) = 0$ has in general no solutions, but under certain conditions an infinite number of solutions.

Before we can introduce these conditions and give the solutions, it is necessary to consider the linear integral equation associated with (1-19), in which the arguments x and t in the kernel function have been interchanged. Then the associated homogeneous equation is of the form

$$v(x) = \lambda_0 \int_a^b K(t,x)v(t)dt \tag{1-40}$$

We will denote by the $\overline{}$ notation the kernel, Fredholm determinant and Fredholm minors of the associated equation, and determine relations between them and the analagous quantities of (1-19).

Let $K(t,x) \equiv \overline{K}(x,t)$. Then the series for $\overline{D}(\lambda)$ becomes

$$\overline{D}(\lambda) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{\lambda^n}{n!} \overline{A}_n$$

But the determinant in $\overline{A}_n$ is the same as that in A_n, except that the rows and columns have been interchanged. This interchange does not alter the value of the determinant, so that $\overline{D}(\lambda) = D(\lambda)$.

The p-th minor of $D(\lambda)$ is given by (1-23). The value of the determinant in the integrand of B_n is the same as that of the determinant in $\overline{B}_n$, since again $\overline{B}_n$ is obtained from B_n by interchanging rows and columns. Therefore

$\bar{B}_n\begin{pmatrix} x_1 & \cdots & x_p \\ y_1 & \cdots & y_p \end{pmatrix} = B_n\begin{pmatrix} y_1 & \cdots & y_p \\ x_1 & \cdots & x_p \end{pmatrix}$, so that

$$\bar{D}\begin{pmatrix} x_1 & \cdots & x_p \\ y_1 & \cdots & y_p \end{pmatrix}\lambda = D\begin{pmatrix} y_1 & \cdots & y_p \\ x_1 & \cdots & x_p \end{pmatrix}\lambda \qquad (1\text{-}41)$$

If λ_o is a characteristic constant of $K(x,t)$ of index q it is also a characteristic constant of $K(t,x)$ of index q. Therefore $\bar{D}_i(x,y;\lambda_o) = D_i(y,x;\lambda_o) = 0$ for $i = 1, \ldots, q-1$, but $\bar{D}_q(x,y;\lambda_o) = D_q(y,x;\lambda_o) \neq 0$. Now if we let $\bar{x}'_\alpha = y'_\alpha$, $\bar{y}'_\alpha = x'_\alpha$ this becomes

$$\bar{D}_q(\bar{x}',\bar{y}';\lambda_o) = D_q(x',y';\lambda_o) \neq 0. \qquad (1\text{-}42)$$

Fredholm's second fundamental theorem now tells us that (1-40) has q linearly independent solutions $\bar{\phi}_\alpha(x)$ given by

$$\bar{\phi}_\alpha(x) = \frac{\bar{D}\begin{pmatrix} \bar{x}'_1 & \cdots, & \bar{x}'_{\alpha-1}, x, \bar{x}'_{\alpha+1} & \cdots & \bar{x}'_q \\ \bar{y}'_1 & \cdots, & \bar{y}'_{\alpha-1}, \bar{y}'_\alpha, \bar{y}'_{\alpha+1} & \cdots & \bar{y}'_q \end{pmatrix}\lambda_o}{\bar{D}\begin{pmatrix} \bar{x}'_1 & \cdots & \bar{x}'_q \\ \bar{y}'_1 & \cdots & \bar{y}'_q \end{pmatrix}\lambda_o} \qquad (1\text{-}43)$$

$$= \frac{D\begin{pmatrix} \bar{y}'_1 & \cdots, & \bar{y}'_{\alpha-1}, \bar{y}'_\alpha, \bar{y}'_{\alpha+1}, & \cdots & \bar{y}'_q \\ \bar{x}'_1 & \cdots, & \bar{x}'_{\alpha-1}, x, \bar{x}'_{\alpha+1}, & \cdots & \bar{x}'_q \end{pmatrix}\lambda_o}{D\begin{pmatrix} \bar{y}'_1 & \cdots & \bar{y}'_q \\ \bar{x}'_1 & \cdots & \bar{x}'_q \end{pmatrix}\lambda_o}$$

But using the above notational change this becomes

$$\bar{\Phi}_{\alpha}(x) = \frac{D\begin{pmatrix} x_1' \cdots x_{\alpha-1}', x_{\alpha}', x_{\alpha+1}' \cdots x_q' \\ y_1' \cdots y_{\alpha-1}', x, y_{\alpha+1}' \cdots y_q' \end{pmatrix}\lambda_o}{D\begin{pmatrix} x_1' \cdots x_q' \\ y_1' \cdots y_q' \end{pmatrix}\lambda_o}$$

Hence the most general solution of (1-40) is

$$v(x) = c_1\bar{\Phi}_1(x) + c_2\bar{\Phi}_2(x) + \cdots c_q\bar{\Phi}_q(x)$$

In a completely analogous way, the function $H(x,y)$ for the associated kernel becomes

$$(1\text{-}44) \qquad \bar{H}(x,y) = \frac{D\begin{pmatrix} y, y_1' \cdots y_q' \\ x, x_1' \cdots x_q' \end{pmatrix}\lambda_o}{D\begin{pmatrix} y_1' \cdots y_q' \\ x_1' \cdots x_q' \end{pmatrix}\lambda_o} \equiv H(y,x)$$

Therefore if we again use the change of notation $\bar{x}_{\alpha}' = y_{\alpha}'$, $\bar{y}_{\alpha}' = x_{\alpha}'$, (1-36) becomes

$$(1\text{-}45) \quad H(t,x) = \lambda_o K(t,x) - \sum_{\alpha=1}^{q} \lambda_o K(t,y_{\alpha}')\,\bar{\Phi}_{\alpha}(x) + \lambda_o\int_a^b H(s,x)K(t,s)ds \,.$$

We are now in a position to discuss the solution of the non-homogeneous integral equation

$$(1\text{-}46) \qquad u(x) = f(x) + \lambda_o\int_a^b K(x,t)u(t)dt$$

for which $D(\lambda_o) = 0$ and λ_o is of index q.

First let us determine the conditions which f(x) must satisfy in order that a solution of (1-46) exist. Suppose that (1-46) is satisfied by some function u(x). Write (1-46) as

$$f(x) = u(x) - \lambda_o \int_a^b K(x,t)u(t)dt$$

Multiply by $\bar{\phi}_\alpha(x)$ and integrate from a to b with respect to x, where $\bar{\phi}_\alpha(x)$ is a solution of (1-40). We then have

$$\int_a^b f(x)\bar{\phi}_\alpha(x)dx = \int_a^b u(x)\bar{\phi}_\alpha(x)dx - \int_a^b \bar{\phi}_\alpha(x)\left[\lambda_o \int_a^b K(x,t)u(t)dt\right]dx$$

$$= \int_a^b u(x)\bar{\phi}_\alpha(x)dx - \int_a^b u(t)\left[\lambda_o \int_a^b K(x,t)\bar{\phi}_\alpha(x)dx\right]dt$$

$$= \int_a^b u(x)\bar{\phi}_\alpha(x)dx - \int_a^b u(t)\bar{\phi}_\alpha(t)dt$$

Thus in order for a solution of (1-46) to exist the f(x) must satisfy

$$\int_a^b f(x)\bar{\phi}_\alpha(x)dx = 0\ , \quad \alpha = 1, \ldots q \tag{1-47}$$

Now suppose that (1-47) is satisfied. Multiplying by $\sum_{\alpha=1}^{q} \lambda_o K(x,y'_\alpha)$, and taking this inside the integral we have

$$\int_a^b \sum_{\alpha=1}^{q} \lambda_o K(x,y'_\alpha)f(t)\bar{\phi}_\alpha(t)dt = 0$$

But from (1-45) this becomes

$$\lambda_o \int_a^b K(x,t)f(t)dt - \int_a^b H(x,t)f(t)dt + \int_a^b f(t)\left[\lambda_o \int_a^b H(s,t)K(x,s)ds\right]dt = 0$$

$$\lambda_o \int_a^b K(x,t)f(t)dt - \int_a^b H(x,t)f(t)dt + \lambda_o \int_a^b \int_a^b H(s,t)K(x,s)f(t)dsdt = 0.$$

In the double integral, interchange the s and t. Therefore

$$\lambda_o \int_a^b K(x,t)f(t)dt - \int_a^b H(x,t)f(t)dt + \lambda_o \int_a^b K(x,t)\left[\int_a^b H(t,s)f(s)ds\right]dt = 0$$

$$\lambda_o \int_a^b K(x,t)\left[f(t) + \int_a^b H(t,s)f(s)ds\right]dt - \int_a^b H(x,t)f(t)dt = 0.$$

Now let $u(t) = f(t) + \int_a^b H(t,s)f(s)ds$. Then $\int_a^b H(x,t)f(t)dt = u(x) - f(x)$.

Hence $\lambda_o \int_a^b K(x,t)u(t)dt - u(x) + f(x) = 0$, or

(1-48) $$u(x) = f(x) + \lambda_o \int_a^b K(x,t)u(t)dt$$

From this we can deduce that if (1-47) is satisfied there exists at least one solution of (1-46).

Now suppose that $v(x)$ is a solution of (1-46), that is

$$v(x) = \lambda_o \int_a^b K(x,t)v(t)dt + f(x)$$

Then $w(x) = u(x) - v(x)$ is a solution of the homogeneous equation

$$w(x) = \lambda_o \int_a^b K(x,t)\left[u(t) - v(t)\right]dt$$

(1-49) $$w(x) = \lambda_o \int_a^b K(x,t)w(t)dt$$

But from (1-34) the most general solution of this equation is of the form

$$c_1 \phi_1(x) + c_2 \phi_2(x) + \dots c_q \phi_q(x)$$

Hence the most general solution of (1-49) is

$$w(x) = u(x) - v(x) = c_1\phi_1(x) + c_2\phi_2(x) + \ldots c_q\phi_q(x)$$

Therefore

$$v(x) = u(x) + c_1\phi_1(x) + c_2\phi_2(x) + \ldots c_q\phi_q(x)$$

and the complete solution of (1-46) is

$$\text{(1-50)}\quad v(x) = f(x) + \int_a^b H(x,t)f(t)dt + c_1\phi_1(x) + \ldots c_q\phi_q(x).$$

We thus arrive at Fredholm's third fundamental theorem:

Theorem 1-3: If λ_o is a characteristic constant of $K(x,t)$ of index q, the equation

$$u(x) = f(x) + \lambda_o\int_a^b K(x,t)u(t)dt$$

has in general no solutions. But an infinite number of solutions of the form (1-50) exist if and only if

$$\int_a^b f(x)\bar{\phi}_\alpha(x)dx = 0\ ,\quad \alpha = 1, \ldots q,$$

where the $\bar{\phi}_i(x)$ are the fundamental functions of the associated homogeneous equation.

CHAPTER II

THE HILBERT THEORY

2.1 DEFINITIONS AND SOME PRELIMINARY RESULTS

Before we can proceed with another discussion of the equation

$$(2\text{-}1) \qquad u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt ,$$

it is necessary to establish a few fundamental results and review some basic definitions.

Definition 1: The inner product of two functions f and g is defined by

$$(f,g) = \int f(x)g(x)dx$$

Definition 2: Two functions, f and g, are said to be orthogonal if $(f,g) = 0$.

Definition 3: The inner product of a function f with itself is called the norm of f, that is

$$(f,f) \equiv Nf = \int (f(x))^2 dx$$

If $Nf = 1$, the function is said to be normalized.

Definition 4: A system of functions $\phi_i(x)$ which are normalized are said to be orthonormal if $(\phi_i, \phi_j) = 0$, $i \neq j$. The orthogonality relationship is then

$$(\phi_i, \phi_j) = \delta_j^i$$

Now let ϕ_i, $i = 1,2, \ldots$, be a system of orthogonal functions, and let f be any function. Define $(f, \phi_j) = c_j$. Obviously

$$\int \left[f(x) - \sum_{j=1}^{n} c_j \phi_j(x)\right]^2 dx \geq 0$$

Then

$$\int f^2(x)dx - 2\sum_{j=1}^{n} c_j \int f(x)\,\phi_j(x)dx + \sum_{j=1}^{n} c_j^2 \int \phi_j^2(x)dx \geq 0.$$

But the ϕ_i are orthogonal, hence $\int \phi_j^2(x)dx = 1$. Therefore

$$\int f^2(x)dx - 2\sum_{j=1}^{n} c_j^2 + \sum_{j=1}^{n} c_j^2 \geq 0$$

$$\int f^2(x)dx \geq \sum_{j=1}^{n} c_j^2$$

The left side of this equation is Nf, and is independent of n, so that

(2-2) $$Nf \geq \sum_{j=1}^{\infty} c_j^2$$

This is known as Bessel's inequality.

We are now in a position to begin a discussion of the solution of the homogeneous equation

(2-3) $$u(x) = \lambda \int_a^b K(x,t)u(t)dt.$$

A value λ for which equation (2-3) has a non-trivial solution is called a characteristic constant of the kernel K(x,t). The linearly independent solutions ϕ_i, $i = 1, 2, \ldots, h$ of (2-3) corresponding to this value of λ are the characteristic functions, and may be assumed normalized and orthogonal, for if not they could be orthogonalized without ceasing to be solutions. Their number, h, for each λ

is finite. To see this we apply Bessel's inequality (2-2) to the kernel $K(x,t)$. Then

$$\lambda^2 \int K^2(x,t)dt \geq \lambda^2 \sum_{i=1}^{\infty} \left[\int K(x,t)\, \phi_i(t)dt \right]^2 .$$

But $\lambda \int K(x,t)\,\phi_i(t)dt = \phi_i(x)$, so that

$$\lambda^2 \int K^2(x,t)dt \geq \sum_{i=1}^{\infty} \phi_i^2(x) \geq \sum_{i=1}^{h} \phi_i^2(x)$$

Now integrate with respect to x, and we have

$$\lambda^2 \int\int K^2(x,t)\; dtdx \geq \sum_{i=1}^{h} \int \phi_i^2(x)dx$$

But since the ϕ_i are orthogonal $\int \phi_i^2(x)dx = 1$, and

$$\lambda^2 \int\int K^2(x,t)dtdx \geq h$$

Thus we have established an upper bound for h.

Finally we show that under certain conditions functions defined by

$$\text{(2-4)} \qquad g(x) = \int_a^b K(x,t)h(t)dt$$

form an equicontinuous and uniformally bounded set (the set of functions g is said to be equicontinuous if for every $\epsilon > 0$ there exists $\delta > 0$, independent of h, such that $|g(x+\eta) - g(x)| < \epsilon$ whenever $|\eta| < \delta$). In (2-4) g is said to be an integral transform of h. If in (2-4) h is piecewise continuous, g is continuous since $K(x,t)$ was assumed continuous. Now suppose $\int h^2(t)dt \leq M^2$, for some fixed M^2. Then the Schwarz inequality yields

$$\left[g(x+\eta) - g(x)\right]^2 = \left[\int_a^b (K(x+\eta,t) - K(x,t))h(t)dt\right]^2$$

$$\leq \int_a^b (K(x+\eta,t) - K(x,t))^2 dt \cdot \int_a^b h^2(t)dt$$

$$< \int_a^b \epsilon^2 dt \cdot M^2 = M^2 \epsilon^2 (b - a),$$

so that $|g(x+\eta) - g(x)| < M\epsilon\sqrt{b-a}$. Hence the functions g defined by (2-4) form an equicontinuous set when $\int h^2(t)dt \leq M^2$.

Again the Schwarz inequality gives us

$$g^2(x) \leq M^2 \int K^2(x,t)dt ,$$

so that the functions g are uniformally bounded.

Thus to integral transforms of the form (2-4) may be applied the following theorem due to Arzela:

Theorem: From any set of uniformally bounded and equicontinuous functions in a domain G, it is possible to pick a subsequence which converges uniformally to a continuous limit function in G.

We are now ready to prove the following theorem:

Theorem 2-1: Either the integral equation

$$\text{(2-5)} \qquad u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt ,$$

~~and consequently the transposed integral equation~~

$$\text{(2-6)} \qquad v(x) = g(x) + \lambda \int_a^b K(t,x)v(t)dt ,$$

with fixed λ, has a unique, continuous solution for each arbitrary continuous function f, or g, (in particular $u = 0$ if $f = 0$, and $v = 0$ if $g = 0$); or the associated homogeneous equations

$$u(x) = \lambda\int_a^b K(x,t)u(t)dt \;, \tag{2-7}$$

$$v(x) = \lambda\int_a^b K(t,x)v(t)dt \tag{2-8}$$

have r linearly independent solutions. In this case the non-homogeneous equation has a solution if and only if

$$(f,v_i) = \int f(x)v_i(x)dx = 0, \; i = 1,\ldots,r.$$

2.2 SOLUTION OF THE EQUATION FOR DEGENERATE KERNELS

A degenerate kernel is one which can be written as a finite sum of products of functions of x and functions of t, that is

$$K(x,t) = \sum_{i=1}^{p} \alpha_i(x)\beta_i(t) \tag{2-9}$$

Making use of (2-9) we can write (2-5) as

$$u(x) = f(x) + \lambda\sum_{i=1}^{p} \alpha_i(x)\int_a^b \beta_i(t)u(t)dt$$

Let $\int \beta_i(t)u(t)dt = X_i$. Then

$$u(x) = f(x) + \lambda\sum_{i=1}^{p} \alpha_i(x)X_i. \tag{2-10}$$

Now multiply by $\beta_j(x)$ and integrate with respect to x.

$$\int u(x)\beta_j(x)dx = \int f(x)\beta_j(x)dx + \lambda\sum_{i=1}^{p} X_i \int \alpha_i(x)\beta_j(x)dx$$

Let $\int f(x)\beta_j(x)dx = f_j$ and $\int \alpha_i(x)\beta_j(x)dx = c_{ij}$. Then

$$X_j = f_j + \lambda\sum_{i=1}^{p} c_{ij}X_i\ , \quad j = 1, 2, \ldots p \tag{2-11}$$

Thus the solution of the integral equation reduces to the solution of a system of p linear equations in p unknowns.

Denote by (a_{ij}) the coefficient matrix of the system (2-11). Then there are two cases.

Case i: $\det(a_{ij}) \neq 0$.

If this determinant is not zero, the system (2-11) can be solved uniquely for the X_j. Therefore (2-10) will be a unique solution of the integral equation (2-5).

However the homogeneous system

$$X_j = \lambda\sum_{i=1}^{p} c_{ij}X_i \tag{2-12}$$

will have only the trivial solution $X_j \equiv 0$, so that the homogeneous integral equation (2-7) will have only the trivial solution $u(x) \equiv 0$.

Next consider the transposed equation (2-6). Make the substitution

$$K(t,x) = \sum_{i=1}^{p} \alpha_i(t)\beta_i(x)$$

and we have

$$v(x) = g(x) + \lambda\sum_{i=1}^{p} \beta_i(x)\int_a^b \alpha_i(t)v(t)dt$$

$$v(x) = g(x) + \lambda \sum_{i=1}^{p} \beta_i(x) Y_i \tag{2-13}$$

Multiply by $\alpha_j(x)$ and integrate with respect to x.

$$\int v(x)\,\alpha_j(x)dx = \int g(x)\,\alpha_j(x)dx + \lambda \sum_{i=1}^{p} Y_i \int \alpha_j(x)\,\beta_i(x)dx$$

$$Y_j = g_j + \lambda \sum_{i=1}^{p} Y_i c_{ji} \;, \; j = 1, \ldots p \tag{2-14}$$

The coefficient matrix of this system is the transpose of that of the system (2-11), and therefore $\det(a_{ji}) \neq 0$. Hence the system (2-14) can be solved uniquely for the Y_j and (2-13) is then a unique solution of (2-6).

The homogeneous system of equations

$$Y_j = \lambda \sum_{i=1}^{p} Y_i c_{ji} \tag{2-15}$$

has as its solution $Y_j \equiv 0$, so that $v(x) \equiv 0$ is the only solution of the transposed homogeneous equation (2-8). Thus under case i, the non-homogeneous equation and its associated transposed equation have unique solutions.

Case ii: $\det(a_{ij}) = 0$.

Now the homogeneous systems (2-12) and (2-15) have an infinite number of non-trivial solutions X_i and Y_i. Therefore equations (2-7) and (2-8) will also have an infinite number of non-trivial solutions, given by

$$u(x) = \lambda \sum_{i=1}^{p} \alpha_i(x) X_i \;, \tag{2-16}$$

$$v(x) = \lambda \sum_{i=1}^{p} \beta_i(x) Y_i \tag{2-17}$$

respectively. But recall the fact, proved in section 2.1, that for each value of λ there are only a finite number of linearly independent characteristic functions. Therefore equations (2-7) and (2-8) have r linearly independent solutions for each characteristic constant λ, in terms of which all other solutions may be expressed.

The non-homogeneous system (2-11), and hence the non-homogeneous equation (2-5) can be solved if and only if the conditions

$$\sum_{j=1}^{p} f_j Y_{kj} = 0 \ , \ k = 1, \ldots r \tag{2-18}$$

are met. But $f_j = \int f(x)\beta_j(x)dx$, so that (2-18) becomes

$$\sum_{j=1}^{p} \left[\int f(x)\beta_j(x)dx \right] Y_{kj} = 0$$

$$\int f(x) \left[\lambda \sum_{j=1}^{p} \beta_j(x) Y_{kj} \right] dx = 0$$

But from (2-17) $\lambda \sum_{j=1}^{p} \beta_j(x) Y_{kj} = v_k(x)$. Therefore (2-5) can be solved if and only if

$$\int f(x) v_k(x) dx = (f, v_k) = 0 \ , \ k = 1, \ldots r$$

We have thus proved our theorem for the case of degenerate kernels.

2.3 SOLUTION OF THE EQUATION FOR ARBITRARY KERNELS

Since every polynomial in x and t represents some degenerate kernel, and since the Weierstrass approximation theorem tells us that every continuous function may be approximated uniformally by polynomials, we suppose that the kernel $K(x,t)$ is uniformally approximated by a sequence of degenerate kernels $A_n(x,t)$. Consider in addition to (2-5) the equations

$$u(x) = f(x) + \lambda \int_a^b A_n(x,t)u(t)dt \tag{2-19}$$

Fix λ. Then there are two cases.

Case i: For infinitely many n, (2-19) has a solution $u_n(x)$ such that $(u_n,u_n) \leq M$, for some fixed M. Then

$$u_n(x) - f(x) = \lambda \int_a^b A_n(x,t)u_n(t)dt$$

and the functions $u_n(x) - f(x)$ are integral transforms of u_n, hence form a uniformally bounded and equicontinuous set. Therefore, from the theorem of Arzela, there exists a subsequence of the functions u_n which converges uniformally to a continuous limit function $\phi(x)$. Then

$$\lim(u_n(x) - f(x)) = \lambda \lim \int_a^b A_n(x,t)u_n(t)dt$$

But since both u_n and A_n converge uniformally this becomes

$$\phi(x) - f(x) = \lambda \int_a^b \lim A_n(x,t) \lim u_n(t)dt$$

$$\phi(x) = f(x) + \lambda \int_a^b K(x,t)\phi(t)dt$$

Therefore equation (2-5) has a solution $\phi(x)$ for every f.

Case ii: Either

a) a solution $u_n(x)$ of (2-19) exists for all n, but $(u_n, u_n) = c_n$ tends to infinity as n tends to infinity, or

b) a solution of (2-19) exists only for a finite number of values of n, in which case we note that

$$u(x) = \lambda \int_a^b A_n(x,t)u(t)dt \tag{2-20}$$

has a solution $\sigma_n(x)$ for every n, by the theorems for degenerate kernels.

In case ii a) divide (2-19) by c_n, and let $\frac{u_n}{c_n} = \sigma_n$. Then

$$\sigma_n(x) - \frac{f(x)}{c_n} = \lambda \int_a^b A_n(x,t)\sigma_n(t)dt$$

In case ii b)

$$\sigma_n(x) = \lambda \int_a^b A_n(x,t)\sigma_n(t)dt$$

In both cases σ_n is normalized. The sequences of integral transforms $\left(\sigma_n(x) - \frac{f(x)}{c_n}\right)$ and $\sigma_n(x)$ both contain subsequences which converge uniformally to functions $\rho(x)$ which satisfy equation (2-7), that is

$$\lim\left(\sigma_n(x) - \frac{f(x)}{c_n}\right) = \lambda \int_a^b \lim(A_n(x,t)\sigma_n(t))dt$$

$$\rho(x) - 0 = \lambda \int_a^b K(x,t)\rho(t)dt$$

and

$$\lim \sigma_n(x) = \lambda \int_a^b \lim(A_n(x,t)\sigma_n(t))dt$$

$$\rho(x) = \lambda \int_a^b K(x,t)\rho(t)dt$$

Therefore under case ii, equation (2-7) always has a non-trivial solution.

Now recall the fact that for each value of λ only a finite number, r, of linearly independent characteristic functions exist. Again there are two cases.

Case i: $r = 0$.

This says that (2-7) has no non-trivial solutions. Thus we are dealing with case i above, since under case ii a non-trivial solution always exists. Therefore the integral equation (2-5) has a solution $\phi(x)$ for every f. Now suppose that $\phi_1(x)$ and $\phi_2(x)$ are two solutions of (2-5). Then

$$\phi_1(x) = f(x) + \lambda\int_a^b K(x,t)\,\phi_1(t)dt$$

$$\phi_2(x) = f(x) + \lambda\int_a^b K(x,t)\,\phi_2(t)dt$$

and subtracting we have

$$\phi_1(x) - \phi_2(x) = \lambda\int_a^b K(x,t)\left[\phi_1(t) - \phi_2(t)\right]dt$$

But since there are no non-trivial solutions of (2-7) by hypothesis, that is $r = 0$, it must be that $\phi_1(x) - \phi_2(x) = 0$, or $\phi_1(x) = \phi_2(x)$, and therefore the solution of (2-5) is unique. Thus if $r = 0$, (2-5) always has a unique solution. Clearly equation (2-6) also has a unique solution.

Case ii: $r > 0$.

Let u_i, $i = 1, \ldots, r$, be the r orthogonal solutions of (2-7). Define

$$\gamma_{ni}(x) = u_i(x) - \lambda\int_a^b A_n(x,t)u_i(t)dt\ ,\quad \begin{array}{l} i = 1, \ldots, r \\ n = 1, 2, \ldots \end{array}$$

Then
$$\lim \gamma_{ni}(x) = \lim u_i(x) - \lambda\int_a^b \lim A_n(x,t)u_i(t)dt$$
$$= u_i(x) - \lambda\int_a^b K(x,t)u_i(t)dt$$
$$= u_i(x) - u_i(x) = 0.$$

Define $A_n'(x,t) = A_n(x,t) + \frac{1}{\lambda}\sum_{i=1}^{r}\gamma_{ni}(x)u_i(x)$. Then

$$\lim A_n'(x,t) = \lim A_n(x,t) + \frac{1}{\lambda}\sum_{i=1}^{r}\lim \gamma_{ni}(x)u_i(x)$$
$$= K(x,t) - 0 = K(x,t).$$

Hence the functions $A_n'(x,t)$ are degenerate kernels which also approximate the kernel $K(x,t)$ uniformally.

We now show that the kernels $A_n'(x,t)$ also possess the r null solutions $u_i(x)$, and no others:

$$\lambda\int_a^b A_n'(x,t)u_j(t)dt = \lambda\int_a^b A_n(x,t)u_j(t)dt + \sum_{i=1}^{r}\gamma_{ni}(x)\int_a^b u_i(t)u_j(t)dt$$
$$= \lambda\int_a^b A_n(x,t)u_j(t)dt + \sum_{i=1}^{r}\gamma_{ni}(x)\,\delta_i^j$$
$$= \lambda\int_a^b A_n(x,t)u_j(t)dt + \gamma_{nj}(x)$$
$$= \lambda\int_a^b A_n(x,t)u_j(t)dt + u_j(x) - \lambda\int_a^b A_n(x,t)u_j(t)dt$$
$$= u_j(x)\ ,\ j = 1, \ldots, r.$$

Therefore the kernels $A_n'(x,t)$ possess the r null solutions $u_i(x)$. Now suppose that in addition $u_{r+1,n}(x)$ is a sequence of null solutions (for sufficiently large n). Then

$$u_{r+1,n}(x) = \lambda\int_a^b A_n'(x,t)u_{r+1,n}(t)dt$$

Let $\lim u_{r+1,n}(x) = u_{r+1}(x)$. Then we have

$$\lim u_{r+1,n}(x) = \lambda\int_a^b \lim(A_n'(x,t)u_{r+1,n}(t))dt$$

$$u_{r+1}(x) = \lambda\int_a^b K(x,t)u_{r+1}(t)dt$$

Hence $u_{r+1}(x)$ is an additional null solution of (2-7), and since $\int u_{r+1}(x)u_k(x) = 0$, $k = 1, \ldots r$, it is orthogonal to $u_1(x), \ldots,$ $u_r(x)$. But this contradicts the hypothesis that (2-7) has exactly r linearly independent solutions. Therefore there cannot exist a sequence of additional null solutions $u_{r+1,n}(x)$, so that the kernels $A_n'(x,t)$ possess exactly the r null solutions $u_i(x)$.

Since our theorem holds for degenerate kernels, the integral equation

$$v(x) = \lambda\int_a^b A_n'(t,x)v(t)dt$$

also has exactly r linearly independent null solutions $v_{i,n}(x)$, $i = 1, \ldots r$, which may be assumed orthonormal. But since the degenerate kernels $A_n'(t,x)$ converge uniformally to $K(t,x)$, Arzela's theorem gives us

$$\lim v_{i,n}(x) = \lambda\int_a^b \lim(A_n'(t,x)v_{i,n}(t))dt$$

$$v_i(x) = \lambda\int_a^b K(t,x)v_i(t)dt \ , \ i = 1, \ldots r.$$

Therefore equation (2-8) also has r linearly independent null solutions $v_i(x)$.

Finally we show that (2-5) has a solution if and only if the conditions $(f,v_i) = 0$, $i = 1, \ldots r$, are met.

Suppose that (2-5) has a solution $\phi(x)$. Then

$$f(x) = \phi(x) - \lambda \int_a^b K(x,t)\phi(t)dt$$

Multiply by $v_i(x)$ and integrate, and we get

$$\int_a^b f(x)v_i(x)dx = \int_a^b \phi(x)v_i(x)dx - \lambda\int_a^b \int_a^b K(x,t)\phi(t)v_i(x)dtdx$$

$$= \int_a^b \phi(x)v_i(x)dx - \int_a^b \left[\lambda\int_a^b K(x,t)v_i(x)dx\right]\phi(t)dt$$

Now make use of (2-8) and we have

$$\int_a^b f(x)v_i(x)dx = \int_a^b \phi(x)v_i(x)dx - \int_a^b v_i(t)\phi(t)dt$$

$$(f,v_i) = 0$$

Now suppose $(f,v_i) = 0$. Let $\lim v_{i,n}(x) = v_i(x)$. Define

$$\epsilon_{in} = \int_a^b f(x)v_{i,n}(x)dx$$

Then

$$\lim \epsilon_{in} = \int_a^b f(x) \lim v_{i,n}(x)dx$$

$$= \int_a^b f(x)v_i(x)dx = 0.$$

Define

$$f_n(x) = f(x) - \sum_{j=1}^{r} \epsilon_{jn}v_{j,n}(x) .$$

Then

$$\int_a^b f_n(x)v_{i,n}(x)dx = \int_a^b f(x)v_{i,n}(x)dx - \sum_{j=1}^{r} \epsilon_{jn} \int_a^b v_{i,n}(x)v_{j,n}(x)dx$$

$$= \int_a^b f(x)v_{i,n}(x)dx - \sum_{j=1}^{r} \epsilon_{jn}\delta_i^j$$

$$= \int_a^b f(x)v_{i,n}(x)dx - \epsilon_{in}$$

$$= \int_a^b f(x)v_{i,n}(x)dx - \int_a^b f(x)v_{i,n}(x)dx$$

$$= 0, i = 1, \ldots r$$

Therefore, since the theorem holds for degenerate kernels, the equation

$$f_n(x) = \phi(x) - \lambda \int_a^b A_n'(x,t)\phi(t)dt$$

has a solution. Note that from the definition of $f_n(x)$,

$$\lim f_n(x) = \lim f(x) - \sum_{i=1}^{r} \lim \epsilon_{in} v_{i,n}(x)$$

$$= f(x) - 0 = f(x).$$

Therefore

$$\lim f_n(x) = \lim \phi(x) - \lambda \int_a^b \lim A_n'(x,t)\phi(t)dt$$

$$f(x) = \phi(x) - \lambda \int_a^b K(x,t)\phi(t)dt,$$

and consequently (2-5) has a solution.

This completes the proof of the theorem for arbitrary kernels.

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VITA

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